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Bounded sets of KKT multipliers in vector optimization

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Abstract In this article we discuss the conditions required to guarantee the non-emptiness and the boundedness of certain subsets of the set of Lagrange multipliers for an inequality and equality constrained vector minimization problem.

1 Introduction and motivation

In this article we will be concerned with the following vector minimization problem (VP)

$$\begin{aligned} \min\ & f(x) = (f_1(x), \ldots, f_m(x)), \\ \text{subject to}\ \ & g_i(x) \leq 0, i = 1, \ldots, k, \\ & h_r(x) = 0, r = 1, \ldots, p, \end{aligned}$$

where $f\colon \mathbb{R}^n \to \mathbb{R}^m$, each $g_i\colon \mathbb{R}^n \to \mathbb{R}$ and each $h_r\colon \mathbb{R}^n \to \mathbb{R}$. We will consider only two different assumptions on the objective and constraint functions. Either we assume that the objective and constraint functions of the problem (VP) are smooth, i.e. continuously differentiable or we assume that all the functions are locally Lipschitz which need not be differentiable. For simplicity in the presentation let us mark the index sets of the objective and constraint functions as follows. Let $M = \{1, \ldots, m\}$, $K = \{1, \ldots, k\}$ and $P = \{1, \ldots, p\}$. It is now well known that there are several solution concepts for a vector minimization problem. Most important among them is the notion of a *Pareto minimum point* or an *efficient point*. Let $\bar{x}$ be a feasible point for (VP). Then $\bar{x}$ is said to be a Pareto minimum if there exists no other feasible point x of (VP) such that $f(x) - f(\bar{x}) \in -(\mathbb{R}^m_+ \setminus \{0\})$. This notion is important both from the theoretical and practical point of view. Another notion called the *weak Pareto minimum* is studied mostly from the theoretical point of view. A point $\bar{x}$ is said to be a weak

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Pareto minimum for the program (VP) if there exists no other feasible x of (VP) such that $f(x) - f(\bar{x}) \in -\text{int}\mathbb{R}^m_+$, where int denotes the interior of a set. It is clear that every Pareto minimum is a weak Pareto minimum but the converse need not be true. Recently there has been quite a few monographs dealing in detail with the mathematics of vector optimization. The interested reader can see for example the recent monographs by Jahn [14], Göpfert et al. [13] and Ehrgott [11] and the references there in. Of course one may also look at the earlier monographs on this subject by Yu [23] and Luc [16].

Consider the program (VP) where the underlying data is smooth. Given a feasible point $\bar{x}$ of (VP) by the term *Karush–Kuhn–Tucker multiplier* (KKT) at $\bar{x}$ we mean a triplet $(\tau, \lambda, \mu) \in \mathbb{R}^m_+ \times \mathbb{R}^k_+ \times \mathbb{R}^p$ with $\tau \neq 0$ such that

(i) $\sum_{j\in M} \tau_j \nabla f_j(\bar{x}) + \sum_{i\in K} \lambda_i \nabla g_i(\bar{x}) + \sum_{r\in P} \mu_r \nabla h_r(\bar{x}) = 0$
(ii) $\lambda_i g_i(\bar{x}) = 0, \quad \forall i \in M.$

For some recent studies on the Lagrange or KKT multipliers for multiobjective optimization problems see for example Ciligot-Travain [7], Craven [9], Amahroq and Taa [1], Maeda [19, 20] and Chandra et al. [5].

The set of all KKT multipliers associated with $\bar{x}$ is denoted as $E(\bar{x})$. If we denote by $I(\bar{x})$ the set of active indices at $\bar{x}$ then set $E(\bar{x})$ is given as follows

$$E(\bar{x}) = \Big\{(\tau, \lambda, \mu) \in (\mathbb{R}^m_+ \setminus \{0\}) \times \mathbb{R}^k_+ \times \mathbb{R}^p \colon \sum_{j\in M} \tau_j \nabla f_j(\bar{x}) + \sum_{i\in I(\bar{x})} \lambda_i \nabla g_i(\bar{x}) + \sum_{r\in P} \mu_r \nabla h_r(\bar{x}) = 0, \lambda_i = 0, i \notin I(\bar{x})\Big\}.$$

A careful look at the set $E(\bar{x})$ will show that for any $\bar{x}$ the set $E(\bar{x})$ is unbounded. Observe that for any $(\tau, \lambda, \mu) \in E(\bar{x})$, the triplet $(\gamma\tau, \gamma\lambda, \gamma\mu) \in E(\bar{x})$ for any real number $\gamma > 0$. On the other hand if we consider the set-valued map $x \mapsto E(x)$ then the graph of E is closed. However the unboundedness of the set of KKT multipliers is not a very desired property. Gauvin [12] first demonstrated that the Mangasarian–Fromovitz constraint qualification (see Mangasarian and Fromovitz [22]) is a necessary and sufficient condition for a mathematical programming problem (scalar optimization) with smooth data to have a bounded set of Lagrange multipliers. Very recently Luskan et al. [17] have demonstrated the importance of the boundedness of the Lagrange multipliers for the study of interior point methods for non-linear programming. Anitescu [2] showed that even if the gradients of the constraints evaluated at the solution point form a linearly dependent set one can develop a good algorithm for constrained optimization if the Mangasarian–Fromovitz constraint qualification holds, i.e. the Lagrange multilpiers or KKT multipliers are bounded. In fact Anitescu [2] designed a sequential quadratic programmimg algorithm using an exact penalty function and showed that one can have nice convergence results under Mangasarian–Fromovitz constraint qualification. Further Anitescu [3] studied mathematical programming problems with only inequality constraints and twice continuously differentiable data which have a non-empty and unbounded Lagrangian multiplier set even though the problem satisfies a quadratic growth condition. Mathematical programming with complementarity constraints or mathematical programming with equilibrium constraints are natural candidates for having unbounded Lagrange multiplier or KKT multiplier sets. This is preciesly due to the fact that for these class of problems the Mangasarian–Fromovitz constraint qualification fails (see for example Dempe [10] or Luo et al. [18]). Anitescu [3] demonstrated that by adding a linear penalty term to the objective function one can transform the problem into an equivalent non-linear programming problem that has a bounded KKT multiplier set.

This motivates us to ask whether in the context of vector optimization we can identify certain subsets of the set $E(\bar{x})$ which are non-empty and bounded under some natural qualification conditions whenever $\bar{x}$ is a Pareto minimum or a weak Pareto minimum for (VP). We shall call such subsets of $E(\bar{x})$ as the set of *proper KKT multipliers* at $\bar{x}$. More precisely we will concentrate on the following subsets of $E(\bar{x})$. For a given $q \in M$ denote by the set $E_q(\bar{x})$ the set of all KKT multipliers for which $\tau_q = 1$. Thus a typical element of $E_q(\bar{x})$ would look as follows

$$(\tau_1, \ldots, \tau_{q-1}, 1, \tau_{q+1}, \ldots, \tau_m, \lambda_1, \ldots, \lambda_k, \mu_1, \ldots, \mu_p).$$

It is important to note that if we multiply an element of $E_q(\bar{x})$ with a scalar $\gamma > 0$ then the resulting element is not an element of $E_q(\bar{x})$ since now $\tau_q = \gamma$. Thus the set $E_q(\bar{x})$ is not closed under scalar multiplication by a positive real number while the set $E(\bar{x})$ is closed under such an operation.

Our main aim in this paper is to investigate the conditions under which the set $E_q(\bar{x})$ is non-empty and bounded. It is intuitive that one has to develop a suitable extension of the Mangasarian–Fromovitz constraint qualification in context of the problem (VP). We will consider the case when the underlying data of the problem (VP) is smooth and also when the underlying data is locally Lipschitz which need not be differentiable. Thus even from the view point of scalar optimization our results will be more general than Gauvin [12]. In fact the sets of the type $E_q(\bar{x})$ will be generically termed as the set of *proper KKT multipliers*. Of course there can be more than one set of non-empty proper KKT multipliers. Chandra et al. [5] studied the boundedness of the sets of proper KKT multipliers when (VP) consists only of inequality constraints and only for the case of a Pareto minimum. Chandra et al. [5] assumed the problem to have locally Lipschitz data. In this article we consider both equality and inequality constraints as well as both Pareto minimum and weak Pareto minimum for smooth and non-smooth cases.

The paper is planned as follows. In Section 2 we discuss the smooth case. We introduce the required qualification condition and prove that there exists a set of proper KKT multipliers at the solution point which is bounded. In Section 2 we only consider Pareto minimum points. In Section 3 we define the regularity condition in the non-smooth setting and consider the boundedness of the sets of proper KKT multipliers at the weak Pareto minimum points. In both the sections we illustrate our results with examples.

2 Smooth case

In this section we assume that the underlying data of the problem (VP) is smooth i.e. the components of the objective function and the constraints are continuously differentiable. We begin with the following qualification condition associated with the problem (VP).

Definition 2.1 Let us consider the problem (VP) with smooth data. Let $\bar{x}$ be a feasible point of (VP). Then the Basic Regularity Condition is said to be satisfied at $\bar{x}$ if there exists $q \in M$ such that the only scalars $\tau_j \geq 0$, $j \in M$, $j \neq q$, $\lambda_i \geq 0$, $i \in I(\bar{x})$, $\lambda_i = 0$, $i \notin I(\bar{x})$, $\mu_r \in \mathbb{R}$, $r \in P$ which satisfy

$$\sum_{j \in M, j \neq q} \tau_j \nabla f_j(\bar{x}) + \sum_{i \in K} \lambda_i \nabla g_i(\bar{x}) + \sum_{r \in P} \mu_r \nabla h_r(\bar{x}) = 0$$

are $\tau_j = 0$, for all $j \in M$, $j \neq q$, $\lambda_i = 0$ for all $i \in K$ and $\mu_r = 0$ for all $r \in P$.

There may be more than one index $q \in M$ for which the above property can be satisfied. However the definition says that we need to just check this for one index. Thus we may also state that the above property as Basic Regularity with respect to the index q. If the *Basic Regularity Condition* (BRC) holds at $\bar{x}$ then it is easy to observe that set of vector $\{\nabla h_1(\bar{x}), \ldots, \nabla h_p(\bar{x})\}$ is linearly independent. Further if BRC holds at a given feasible point $\bar{x}$ of (VP) then $p < n$. Assume on the contrary that BRC holds at $\bar{x}$ and $p = n$. Since $\{\nabla h_1(\bar{x}), \ldots, \nabla h_p(\bar{x})\}$ is linearly independent the only solution for the system

$$\langle \nabla h_r(\bar{x}), d\rangle = 0, \quad \forall r \in P$$

is $d = 0$. Thus for any $q \in M$ the following system

$$\begin{aligned} \langle \nabla f_j(\bar{x}), d\rangle &< 0, \qquad j \in M, j \neq q, \\ \langle \nabla g_i(\bar{x}), d\rangle &< 0, \qquad i \in I(\bar{x}), \\ \langle \nabla h_r(\bar{x}), d\rangle &= 0, \qquad r \in P \end{aligned}$$

has no solution.
Thus using Motzkin Theorem of the Alternative (see Mangasarian [21] Chapter 2) one can show that there exists scalars $\tau_j \geq 0$, $j \neq q$, $\lambda_i \geq 0$, $i \in I(\bar{x})$, not all zero and $\mu_r \in \mathbb{R}$, $r \in P$ such that

$$\sum_{j\in M, j\neq q} \tau_j \nabla f_j(\bar{x}) + \sum_{i\in K} \lambda_i \nabla g_i(\bar{x}) + \sum_{r\in P} \mu_r \nabla h_r(\bar{x}) = 0.$$

This clearly contradicts the fact that BRC holds at $\bar{x}$.

Theorem 2.1 *Let us consider the problem (VP) with smooth data. Let $\bar{x}$ be a Pareto minimum for (VP). Further assume that BRC holds at the point $\bar{x}$. Then there exists a set of proper KKT multipliers which is non-empty and bounded.*

Proof Since BRC holds at $\bar{x}$ there exists $q \in M$ such that $\tau_j \geq 0$, $j \in M$, $j \neq q$, $\lambda_i \geq 0$, $i \in I(\bar{x})$, $\lambda_i = 0$, $i \notin I(\bar{x})$, $\mu_r \in \mathbb{R}$, $r \in P$ with

$$\sum_{j\in M, j\neq q} \tau_j \nabla f_j(\bar{x}) + \sum_{i\in K} \lambda_i \nabla g_i(\bar{x}) + \sum_{r\in P} \mu_r \nabla h_r(\bar{x}) = 0,$$

implies that $\tau_j = 0$, for all $j \in M$, $j \neq q$, $\lambda_i = 0$ for all $i \in K$ and $\mu_r = 0$ for all $r \in P$. On the other hand since $\bar{x}$ is a Pareto minimum point we know from Chankong and Haimes [4] that $\bar{x}$ is a solution to the following problem

$$\begin{aligned} \min\ & f_q(x), \\ \text{subject to}\ & f_j(x) \leq f_j(\bar{x}), j \in M, j \neq q, \\ & g_i(x) \leq 0, i \in K, \\ & h_r(x) = 0, r \in P. \end{aligned}$$

Thus by the Fritz John conditions we have scalars $\tau_q \geq 0$, $\tau_j \geq 0$, $j \in M$, $j \neq q$, $\lambda_i \geq 0$, $i \in K$ and $\mu_r \in \mathbb{R}$, $r \in P$ with

$$\tau_q \nabla f_q(\bar{x}) + \sum_{j\in M, j\neq q} \tau_j \nabla f_j(\bar{x}) + \sum_{i\in K} \lambda_i \nabla g_i(\bar{x}) + \sum_{r\in P} \mu_r \nabla h_r(\bar{x}) = 0, \tag{1}$$

and

$$\lambda_i g_i(\bar{x}) = 0, \qquad \forall i \in K.$$

Since BRC holds at $\bar{x}$ it is immediate that $\tau_q > 0$ and without loss of generality we can consider $\tau_q = 1$. Hence we have

$$(\tau_1, \ldots, \tau_{q-1}, 1, \tau_{q+1}, \ldots, \tau_m, \lambda_1, \ldots, \lambda_k, \mu_1, \ldots, \mu_p) \in E_q(\bar{x}).$$

Thus $E_q(\bar{x})$ is non-empty. We now claim that $E_q(\bar{x})$ is bounded. On the contrary assume that it is unbounded. Observe that we can write (1) in a compact manner as

$$\nabla f_q(\bar{x}) + JF_q(\bar{x})^T y = 0, \tag{2}$$

where

$$F_q(\bar{x}) = (f_1(\bar{x}), \ldots, f_{q-1}(\bar{x}), f_{q+1}(\bar{x}), \ldots, f_m(\bar{x}), g_1(\bar{x}), \ldots, g_k(\bar{x}), h_1(\bar{x}), \ldots, h_p(\bar{x})),$$

and $JF_q(\bar{x})$ denotes the Jacobian matrix of F at $\bar{x}$ with T denoting the transpose of the matrix and the vector y is given as

$$y = (\tau_1, \ldots, \tau_{q-1}, \tau_{q+1}, \ldots, \tau_m, \lambda_1, \ldots, \lambda_k, \mu_1, \ldots, \mu_p).$$

If $E_q(\bar{x})$ is unbounded then one can find a sequence $\{y_s\}$ where y_s is given as

$$y_s = (\tau_1^s, \ldots, \tau_{q-1}^s, \tau_{q+1}^s, \ldots, \tau_m^s, \lambda_1^s, \ldots, \lambda_k^s, \mu_1^s, \ldots, \mu_p^s)$$

such that $\|y_s\| \to +\infty$. Now consider the sequence $\{v_s\}$ with $v_s = \dfrac{y_s}{\|y_s\|}$. Hence from (2) we have for each s

$$0 = \frac{1}{\|y_s\|} \nabla f_q(\bar{x}) + JF_q(\bar{x})^T v_s.$$

Since $\{v_s\}$ is bounded it has a convergent subsequence converging to say $v_0 \neq 0$. Thus in the limit we have

$$0 = JF_q(\bar{x})^T v_0.$$

This clearly contradicts the fact that BRC holds at $\bar{x}$ with respect to q. Hence the result. □

Remark 2.1 It is clear from the above result that BRC is a natural qualification condition for a Pareto minimum point. If $\bar{x}$ is a Pareto minimum then the Chankong and Haimes [4] scalarization criteria tells us that $\bar{x}$ solves all problems of the form

$$\begin{aligned} &\min\ f_q(x), \\ \text{subject to}\quad & f_j(x) \leq f_j(\bar{x}), j \in M, j \neq q, \\ & g_i(x) \leq 0, i \in K, \\ & h_r(x) = 0, r \in P, \end{aligned}$$

where q varies over M. Thus there are m scalar subproblems associated with the Chankong and Haimes [4] criteria. From the discussion following the definition of BRC it is clear that if $\bar{x}$ is a Pareto minimum point of (VP) and if BRC holds $\bar{x}$ then there is at least one scalar subproblem of the above form for which the Mangasarian–Fromovitz constraint qualification [22] holds. Moreover the converse is also true, i.e. if there is a scalar subproblem of the above form for which the Mangasarian–Fromovitz constraint qualification holds at $\bar{x}$ then BRC holds for (VP) at $\bar{x}$. In the case when (VP) is a scalar problem i.e. $m = 1$ the above theorem provides a much simpler proof than that of Gauvin [12].

However the existence of a proper KKT multiplier for (VP) at the Pareto minimum point can in fact be guaranteed by a condition much weaker than that of BRC. We shall call this as the *Basic Constraint Qualification* (BCQ). The problem (VP) is said to satisfy the BCQ at a feasible point $\bar{x}$ if the only scalars $\lambda_i \geq 0, i \in I(\bar{x}), \lambda_i = 0, i \notin I(\bar{x})$ and $\mu_r \in \mathbb{R}, r \in P$ which satisfy

$$\sum_{i \in K} \lambda_i \nabla g_i(\bar{x}) + \sum_{r \in P} \mu_r \nabla h_r(\bar{x}) = 0$$

are $\lambda_i = 0, i \in K$ and $\mu_r = 0$, $r \in P$.
It is clear that if BRC holds at $\bar{x}$ then BCQ holds at $\bar{x}$ too but the converse need not be true. Thus BCQ is a weaker qualification condition than BRC. We provide below two examples. The first example demonstrate that if BRC holds then we indeed have a set of proper KKT multipliers which is bounded. The second example shows that all the sets of proper KKT multipliers will be unbounded if BRC fails to hold even though BCQ holds.

Example 2.1 Consider the following vector optimization problem,

$$\min(f_1(x), f_2(x)), \quad \text{subject to} \quad g(x) \leq 0, \quad h(x) = 0,$$

where f_1, f_2, g and h are real-valued functions on R^3. The functions are given as follows

$$\begin{aligned} f_1(x_1, x_2, x_3) &= x_1^3 + x_3, \\ f_2(x_1, x_2, x_3) &= -x_3, \\ g(x_1, x_2, x_3) &= -x_1, \\ h(x_1, x_2, x_3) &= x_1^3 + 2x_2. \end{aligned}$$

It is clear that (0, 0, 0) is an efficient point for the above vector program.
Consider the index $j = 1$. Let us now consider the expression

$$\tau_2 \nabla f_2(0,0,0) + \lambda \nabla g(0,0,0) + \mu \nabla h(0,0,0) = 0,$$

where $\tau_2 \geq 0$, $\lambda \geq 0$ and $\mu \in R$. It is easy to show that the above equation holds only for $\tau_2 = 0, \lambda = 0$ and $\mu = 0$. Hence the BRC holds at (0, 0, 0).

Now let us consider the set of proper KKT multipliers $E_1((0,0,0))$ given as

$$\begin{aligned} E_1((0,0,0)) &= \{(1, \tau_2, \lambda, \mu) \colon (0,0,1) + \tau_2(0,0,-1) + \lambda(-1,0,0) + \mu(0,2,0) = 0\} \\ &= \{(1,1,0,0)\}. \end{aligned}$$

Thus it is clear that $E_1((0,0,0))$ is bounded.

Example 2.2 Consider the following vector optimization problem,

$$\min(f_1(x), f_2(x)), \quad \text{subject to} \quad g(x) \leq 0, \quad h(x) = 0,$$

where f_1, f_2, g and h are real-valued functions on R^2 which are given as

$$\begin{aligned} f_1(x) &= x_1^2 + x_2^2, \\ f_2(x) &= -2x_1 x_2, \\ g(x) &= x_1, \\ h(x) &= x_1 - x_2. \end{aligned}$$

Observe that the feasible set of this problem is the set $\{(u, u) \in \mathbb{R}^2 : u \leq 0\}$. Further observe that every feasible point is a Pareto minimum. Let us consider the point $\bar{x} = (0, 0)$. We will first prove that BRC fails at (0, 0).

Consider $j = 1$ and let

$$0 = \tau_2 \nabla f_2(0,0) + \lambda \nabla g(0,0) + \mu \nabla h(0,0)$$
$$= \tau_2(0,0) + \lambda(1,0) + \mu(1,-1).$$

Then it is easy to observe that the above relation holds for all $\tau_2 \geq 0, \lambda = 0, \mu = 0$.

Now consider $j = 2$ and

$$0 = \tau_1 \nabla f_1(0,0) + \lambda \nabla g(0,0) + \mu \nabla h(0,0)$$
$$= \tau_1(0,0) + \lambda(1,0) + \mu(1,-1).$$

Again it is easy to observe that the above relation holds for $\tau_1 \geq 0, \lambda = 0, \mu = 0$. Thus it is clear that BRC fails at $(0,0)$. However observe that BCQ holds at $(0,0)$ since the following expression

$$0 = \lambda \nabla g(0,0) + \mu \nabla h(0,0)$$
$$= \lambda(1,0) + \mu(1,-1),$$

holds only for $\lambda = 0$ and $\mu = 0$.
Further it is easy to calculate the set of proper KKT multipliers at $(0,0)$ which are given as

$$E_1((0,0)) = \{(1, \tau_2, \lambda, \mu) : \tau_2 \geq 0, \lambda = 0, \mu = 0\}$$

and

$$E_2((0,0)) = \{(\tau_1, 1, \lambda, \mu) : \tau_1 \geq 0, \lambda = 0, \mu = 0\}.$$

Hence it is clear that the set of proper KKT multipliers are unbounded.

The above example in fact illustrates the following result whose simple proof is omitted.

Theorem 2.2 *Let us consider the smooth problem (VP) with smooth data. Let $\bar{x}$ be a feasible point of (VP) for which there exists a set of proper KKT multipliers which is non-empty and bounded then BRC holds at $\bar{x}$.*

3 Non-smooth extensions

Throughout this section we will consider the problem (VP) with locally Lipschitz data, i.e we will consider that the components of the objective function and the constraints are locally Lipschitz. Our main tool in this section will be the Clarke generalized derivative and the Clarke subdifferential of a locally Lipschitz function. For any given locally Lipschitz function $\phi : \mathbb{R}^n \to \mathbb{R}$ and any $x \in \mathbb{R}^n$ the Clarke generalized derivative at x in the direction v is denoted as $\phi^\circ(x, v)$ and the Clarke subdifferential of ϕ at x is denoted as $\partial^\circ \phi(x)$. For details on the Clarke generalized derivative and the Clarke subdifferential see Clarke [8].

Now consider the problem (VP) with locally Lipschitz data and let $\bar{x}$ be a feasible point for (VP). Then by KKT multipliers of (VP) at $\bar{x}$ we mean the triplet $(\tau, \lambda, \mu) \in \mathbb{R}^m_+ \times \mathbb{R}^k_+ \times \mathbb{R}^p$ with $\tau \neq 0$ such that

$$0 \in \sum_{j \in M} \tau_j \partial^\circ f_j(\bar{x}) + \sum_{i \in K} \lambda_i \partial^\circ g_i(\bar{x}) + \sum_{r \in P} \mu_r \partial^\circ h_r(\bar{x}) \tag{3}$$

and

$$\lambda_i g_i(\bar{x}) = 0 \qquad \forall i \in K. \tag{4}$$

Thus the set of KKT multipliers $E(\bar{x})$ for the problem (VP) consists of all triplets $(\tau, \lambda, \mu) \in \mathbb{R}^m_+ \times \mathbb{R}^k_+ \times \mathbb{R}^p$ with $\tau \neq 0$ such that (3) and (4) hold. For $q \in M$ we shall denote by $E_q(\bar{x})$ the subset of $E(\bar{x})$ whose every element has $\tau_q = 1$. As before even in the context of non-smooth vector optimization we will generically term the sets $E_q(\bar{x})$ as the set of *proper KKT multipliers* of (VP) at $\bar{x}$. As in the previous section our aim in this section is to define a qualification condition which would guarantee the boundedness of the sets of proper KKT multipliers. Apart from considering Pareto minimum points, in this section we shall also look at the sets of proper KKT multipliers at weak Pareto minimum points.

Definition 3.1 Let us consider the problem (VP) with locally Lipschitz data. Let $\bar{x}$ be feasible point of (VP). Then the Basic Regularity Condition (BRC) is said to be satisfied at $\bar{x}$ if there exists $q \in M$ such that the only scalars $\tau_j \geq 0$, $j \in M$, $j \neq q$, $\lambda_i \geq 0$, $i \in I(\bar{x})$, $\lambda_i = 0$, $i \notin I(\bar{x})$, $\mu_r \in \mathbb{R}$, $r \in P$ which satisfy

$$0 \in \sum_{j \in M, j \neq q} \tau_j \partial^\circ f_j(\bar{x}) + \sum_{i \in K} \lambda_i \partial^\circ g_i(\bar{x}) + \sum_{r \in P} \mu_r \partial^\circ h_r(\bar{x}),$$

are $\tau_j = 0$, for all $j \in M$, $j \neq q$, $\lambda_i = 0$ for all $i \in K$ and $\mu_r = 0$ for all $r \in P$.

The definition of BRC for a the problem (VP) with locally Lipschitz data was introduced in Chandra et al. [5]. For more details on various constraint qualifications and regularity conditions in multiobjective optimization see for example Li [15].

To motivate the results in this section we begin with an example showing that if BRC fails at a Pareto minimum point, then every set of proper KKT multipliers can be unbounded.

Example 3.1 Consider the following vector optimization problem,

$$\min(f_1(x), \ f_2(x)), \quad \text{subject to} \quad g(x) \leq 0, \quad h(x) = 0,$$

where f_1, f_2, g and h are real-valued functions on $\mathbb{R}^2$.

$$f_1(x_1, x_2) = \begin{cases} x_1 & : x_1 \geq 0, \\ x_1^2 + x_2^2 & : x_1 < 0. \end{cases} \quad f_2(x_1, x_2) = \begin{cases} x_2^3 & : x_2 \geq 0, \\ 0 & : x_2 < 0. \end{cases}$$

$$g(x_1, x_2) = \begin{cases} x_1 + x_2 & : x_1 + x_2 \geq 0, \\ x_1 + 1 & : x_1 + x_2 < 0. \end{cases}$$

and

$$h(x_1, x_2) = x_1 - x_2.$$

Note that the point $\bar{x} = (0, 0)$ is an efficient point. The Clarke subdifferentials of the functions are given as follows

$$\begin{aligned} \partial^\circ f_1(0,0) &= \{(\xi_1, \xi_2) : 0 \leq \xi_1 \leq 1, \xi_2 = 0\}, \\ \partial^\circ f_2(0,0) &= \{(0,0)\}, \\ \partial^\circ g(0,0) &= \{(\xi_1, \xi_2) : \xi_1 = 1, 0 \leq \xi_2 \leq 1\}, \\ \partial^\circ h(0,0) &= \{(1, -1)\}. \end{aligned}$$

We will now demonstrate that the BRC fails at $(0, 0)$. Observe first that g is active at $(0, 0)$. Now consider $j = 1$. Take any $\tau_2 > 0$, $\lambda = 0$ and $\mu = 0$. Then observe that

$$(0, 0) = \tau_2(0, 0) + \lambda(1, 1) + \mu(1, -1) \in \tau_2\partial^\circ f_2(0, 0) + \lambda\partial^\circ g(0, 0) + \mu\partial^\circ h(0, 0).$$

Now consider $j = 2$. Take any $\tau_1 > 0$, $\lambda = 0$ and $\mu = 0$. Then observe that

$$(0, 0) = \tau_1(0, 0) + \lambda(1, 1) + \mu(1, -1) \in \tau_1\partial^\circ f_1(0, 0) + \lambda\partial^\circ g(0, 0) + \mu\partial^\circ h(0, 0).$$

Hence it is clear that BRC does not hold. Now it is easy to see that

$$E_1((0, 0)) = \{(1, \tau_2, \lambda, \mu) \colon \tau_2 \geq 0, \lambda = 0, \mu = 0\}$$

and

$$E_2((0, 0)) = \{(\tau_1, 1, \lambda, \beta) \colon \tau_1 \geq 0, \lambda = 0, \mu = 0\}.$$

Hence all the proper KKT multiplier sets are unbounded.

Now we will turn to the case of the weak Pareto minimum by assuming that the underlying data in (VP) is locally Lipschitz.

Theorem 3.1 *Let us consider the problem (VP) where each f_j, $j \in M$, each g_i, $i \in K$ and each h_r, $r \in P$ are locally Lipschitz functions. Let $\bar{x}$ be a weak Pareto minimum for (VP). Assume that the Basic Regularity Condition holds at $\bar{x}$. Then there exists a set of proper KKT multipliers for (VP) at $\bar{x}$ which is non-empty and bounded.*

Proof Since $\bar{x}$ is a weak minimum for (VP) then it is easy to show that $\bar{x}$ solves the following scalar problem

$$\min F(x) \quad \text{subject to} \quad h_r(x) = 0, \quad r \in P,$$

where $F(x) = \max\{f_1(x) - f_1(\bar{x}), \ldots, f_m(x) - f_m(\bar{x}), g_1(x), \ldots, g_k(x)\}$. If not then there exists $y \in \mathbb{R}^n$ such that $F(y) - F(\bar{x}) < 0$ and $h_r(y) = 0$ for all $r \in P$. Further observe that since $\bar{x}$ is a weak Pareto minimum for (VP) then $g_i(\bar{x}) \leq 0$ for all $i \in K$. This shows that $F(\bar{x}) = 0$. Hence we have $F(y) < 0$. This shows that y is feasible for (VP) and thus the weak Pareto minimality of $\bar{x}$ is contradicted. Hence using Theorem 6.2.1 in Clarke [8] we have that there exist scalars $\tau_0 \geq 0$ and $\mu_r \in \mathbb{R}^p$, not all zero such that

$$0 \in \tau_0\partial^\circ F(\bar{x}) + \sum_{r \in P} \mu_r\partial^\circ h_r(\bar{x}). \tag{5}$$

From Proposition 2.3.12 in Clarke [8] we have that

$$\partial^\circ F(\bar{x}) \subseteq \text{co}\left\{\left(\bigcup_{j \in M} \partial^\circ f_j(\bar{x})\right) \bigcup \left(\bigcup_{i \in I(\bar{x})} \partial^\circ g_i(\bar{x})\right)\right\}.$$

Thus using (5) we can conclude that there exist scalars $\tau_j \geq 0$, $j \in M$, $\lambda_i \geq 0$, $i \in K$ and $\mu_r \in \mathbb{R}$ such that

$$0 \in \sum_{j \in M} \tau_j\partial^\circ f_j(\bar{x}) + \sum_{i \in K} \lambda_i\partial^\circ g_i(\bar{x}) + \sum_{r \in P} \mu_r\partial^\circ h_r(\bar{x}) \tag{6}$$

and

$$\lambda_i g_i(\bar{x}) = 0, \quad \forall i \in K.$$

Now since BRC holds there exists an index $q \in M$ such that the only scalars $\tau_j \geq 0$, $j \in M$, $j \neq q$, $\lambda_i \geq 0, i \in I(\bar{x}), \lambda_i = 0, i \notin I(\bar{x}), \mu_r \in \mathbb{R}, r \in P$ which satisfy

$$0 \in \sum_{j \in M, j \neq q} \tau_j \partial^\circ f_j(\bar{x}) + \sum_{i \in K} \lambda_i \partial^\circ g_i(\bar{x}) + \sum_{r \in P} \mu_r \partial^\circ h_r(\bar{x}),$$

are $\tau_j = 0$, for all $j \in M$, $j \neq q$, $\lambda_i = 0$ for all $i \in K$ and $\mu_r = 0$ for all $r \in P$.
This clearly shows that $\tau_q > 0$ and we can consider $\tau_q = 1$ and thus $E_q(\bar{x}) \neq \emptyset$. Further we claim that $E_q(\bar{x})$ is bounded. On the contrary let us assume that the $E_q(\bar{x})$ is unbounded. Then we have a sequence of vectors $\{\theta_s\}$, with $\theta_s \in E_q(\bar{x})$ of the form

$$\theta_s = (\tau_1^s, \ldots, \tau_{q-1}^s, 1, \tau_{q+1}^s, \ldots, \tau_m^s, \lambda_1^s, \ldots, \lambda_k^s, \mu_1^s, \ldots, \mu_p^s)$$

with $\|\theta_s\| \to +\infty$ as $s \to +\infty$. Thus from (6) we have

$$0 \in \partial^\circ f_q(\bar{x}) + \sum_{j \in M, j \neq q} \tau_j^s \partial^\circ f_j(\bar{x}) + \sum_{i \in K} \lambda_i^s \partial^\circ g_i(\bar{x}) + \sum_{r \in P} \mu_r^s \partial^\circ h_r(\bar{x}).$$

Consider the sequence $w_s = \dfrac{\theta_s}{\|\theta_s\|}$. Thus we have

$$0 \in \frac{1}{\|\theta_s\|} \partial^\circ f_q(\bar{x}) + \sum_{j \in M, j \neq q} \frac{\tau_j^s}{\|\theta_s\|} \partial^\circ f_j(\bar{x}) + \sum_{i \in K} \frac{\lambda_i^s}{\|\theta_s\|} \partial^\circ g_i(\bar{x}) + \sum_{r \in P} \frac{\mu_r^s}{\|\theta_s\|} \partial^\circ h_r(\bar{x}).$$

Since $\{w_s\}$ is a bounded sequence there exists a convergent subsequence which converges to say $w^* \neq 0$. Then noting that the Clarke subdifferentials are compact sets we have in the limit

$$0 \in \sum_{j \in M, j \neq q} \tau_j^* \partial^\circ f_j(\bar{x}) + \sum_{i \in K} \lambda_i^* \partial^\circ g_i(\bar{x}) + \sum_{r \in P} \mu_r^* \partial^\circ h_r(\bar{x})$$

and we also have

$$w^* = (\tau_1^*, \ldots, \tau_{q-1}^*, 0, \tau_{q+1}^*, \ldots, \tau_m^*, \lambda_1^*, \ldots, \lambda_k^*, \mu_1^*, \ldots, \mu_p^*).$$

This is clearly a contradiction to the fact that BRC holds with respect to the index q. Hence the result.
Let us now illustrate the above theorem through the following example

Example 3.2 Consider the following vector optimization problem,

$$\min(f_1(x), f_2(x)), \quad \text{subject to} \quad g(x) \leq 0, \quad h(x) = 0,$$

where f_1, f_2, g and h are real-valued functions on $\mathbb{R}^2$ and are defined as follows.

$$f_1(x_1, x_2) = x_1 + x_2,$$

$$f_2(x_1, x_2) = \begin{cases} x_1 + x_2^2 & : x_1 + x_2 \geq 0, \\ 0 & : x_1 + x_2 < 0. \end{cases}$$

$$g(x_1, x_2) = -1 - x_1$$

and

$$h(x_1, x_2) = \begin{cases} x_1 + \frac{1}{2} x_2 & : x_1 \geq 0, \\ x_1 + x_2^2 & : x_1 < 0. \end{cases}$$

The point $\bar{x} = (0, 0)$ is a weak Pareto minimum for the above problem but not a Pareto minimum. The Clarke subdifferential of the functions associated with the above problem is as follows

$$\begin{aligned} \partial^\circ f_1(0,0) &= \{(1,1)\}, \\ \partial^\circ f_2(0,0) &= \{(\xi_1, \xi_2) : 0 \le \xi_1 \le 1, \xi_2 = 0\}, \\ \partial^\circ g(0,0) &= \{(-1,0)\}, \\ \partial^\circ h(0,0) &= \{(\xi_1, \xi_2) : \xi_1 = 1, 0 \le \xi_2 \le \tfrac{1}{2}\}. \end{aligned}$$

Let us first show that BRC holds at $\bar{x} = (0, 0)$. Consider the index $j = 2$. Observe that g is inactive at $(0, 0)$. Then the following expression

$$0 \in \tau_1 \partial^\circ f_1(0,0) + \mu \partial^\circ h(0,0), \quad \tau_1 \ge 0,$$

only holds for $\tau_1 = 0$ and $\mu = 0$.
Further observe that the set of proper KKT multipliers $E_2((0, 0))$ is given as

$$E_2((0,0)) = \bigcup_{0 \le t \le 1} \{(\tau_1, 1, 0, \mu) : 0 \le \tau_1 \le t, -2t \le \mu \le -t\}.$$

Thus it is clear that for each $(\tau_1, 1, 0, \mu) \in E_2((0, 0))$ one has $0 \le \tau_1 \le 1$ and $-2 \le \mu \le 0$. Thus $E_2((0, 0))$ is bounded.

Consider $\bar{x}$ to be a Pareto minimum of the problem (VP) with locally Lipschitz data at which BRC holds. Then by using the Chankong and Haimes [4] scalarization criteria and Theorem 6.2.1 in Clarke [8] we can easily show the existence of proper KKT multipliers. Then by using a similar approach as Theorem 3.1 we can establish the boundedness of a set of proper KKT multipliers. This is summed up in the following theorem

Theorem 3.2 *Let us consider the program (VP) with locally Lipschitz data. Assume that $\bar{x}$ is a Pareto minimum of (VP). Further assume that BRC holds at $\bar{x}$. Then there exists a set of proper KKT multipliers which is non-empty and bounded.*

Remark 3.1 It is important to note that in order to prove the existence of proper KKT multipliers one can consider a condition which is weaker than BRC and even in the non-smooth setting we shall refer to it as the Basic Constraint Qualification (BCQ). This is given as follows. The problem (VP) is said to satisfy BCQ at $\bar{x}$ if the only scalars $\lambda_i \ge 0$, $i \in I(\bar{x})$ and μ_r, $r \in P$ which satisfy

$$0 \in \sum_{i \in I(\bar{x})} \lambda_i \partial^\circ g_i(\bar{x}) + \sum_{r \in P} \mu_r \partial^\circ h_r(\bar{x})$$

are $\lambda_i = 0, i \in K$ and $\mu_r = 0, r \in P$.

It is easy to check that in Example 3.1, BCQ holds at $\bar{x}$ even though BRC fails to hold and all the sets of proper KKT multipliers are unbounded. Thus BRC seems to take on a fundamental role in vector optimization since it appears to be the only condition which guarantees the non-emptiness and boundedness of a set of proper KKT multipliers at the same time irrespective of the fact whether the solution is a Pareto minimum or a weak Pareto minimum. This role of BRC remains unchanged as we move from smooth to non-smooth optimization. Further the importance of BRC mainly stems from the fact that it is an essential tool to analyze even the weak Pareto optimal point for which there is no Chankong and Haimes [4] type scalarization scheme in general.

Remark 3.2 It is interesting to note that by using a scalarization scheme due to Charnes and Cooper [6] recently Zlobec [24] has derived a necessary and sufficient optimality condition for the existence of a Pareto minimum for the problem (VP) when (VP) is a non-smooth convex vector optimization problem consisting of only inequality constraints (see [24] Chapter 6, Theorem 6.3). Zlobec [24] assumes that each component of the objective function and the constraints which are active at the Pareto minimum point satisfies the *locally flat surface* (LFS) property at the Pareto minimum point. For more details on the LFS property see [24]. The LFS property acts as a regularity condition in the case of a convex vector optimization. However observe that in Theorem 6.3 in [24] one needs to assume the LFS property on all the components of the objective function while for the BRC to hold we for the problem (VP) we need $m - 1$ components of the objective function, i.e we need all the components of the objective function except one.

The following theorem says that the failure of BRC will lead to the unboundedness of the sets of proper KKT multipliers. The proof is simple and hence we omit that.

Theorem 3.3 *Let us consider the program (VP) with locally Lipschitz data. Consider $\bar{x}$ to be a feasible point for (VP). Let there exist a non-empty and bounded set of proper KKT multipliers at $\bar{x}$. Then (VP) satisfies BRC at $\bar{x}$.*

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